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A NOTE ON NORM INEQUALITIES FOR INTEGRAL OPERATORS ON CONES

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ABSTRACT. Norm inequalities for the Riemann-Liouville operator $R_r f(x) = \int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)f(t)dt$ and Weyl operator $W_r f(x) = \int_{\langle x,\infty\rangle} \Delta_V^{r-1}(t-x)f(t)dt$ on cones in R^d have been obtained in the case $r \geq 1$ [7]. In this note, these inequalities are further extended to the case $r < 1$. The question of whether the Hardy operator $Hf(x) = \int_{\langle 0,x\rangle} f(t)dt$ on cones is bounded from $L^p(\Delta_V^\alpha(x))$ to $L^q(\Delta_V^\beta(x))$ $(q < p)$ is also solved.

Let V be a homogeneous cone in R^d. V defines a partial ordering in R^d in such a way that $x <_V y$ if and only if $y - x \in V$. The cone interval $\langle a, b\rangle$ is thus given by $\langle a, b\rangle = \{x \in V: a <_V x <_V b\}$. For $x \in V$ we define $\Delta_V(x) = \int_{\langle 0,x\rangle} dy$.

Let $G(V)$ denote the automorphism group of V, and let $\Sigma = \{x \in V: |x| = 1\}$, $\sigma_0 = \sigma_0(V) = \inf\{\alpha: \int_\Sigma \Delta^\alpha(t')dt' < \infty\}$ and $\sigma(V) = \max(-1, \sigma_0)$. It is known (see [4, 7]) that if $\alpha > \sigma(V)$, then $\int_{\langle 0,x\rangle} \Delta_V^\alpha(t)dt$ is finite for all $x \in V$ and homogeneous of order $\alpha + 1$ so that

$$\int_{\langle 0,x\rangle} \Delta_V^\alpha(t)\,dt = c\Delta_V^{\alpha+1}(x).$$

The dual V^* of V is defined as $V^* = \{x \in R^d: x \cdot y > 0, \forall y \in \overline{V}, y \neq 0\}$. Clearly, V^* is also a cone. It is known that $V^{**} = V$.

The $*$-function on V is the mapping $x \to x^*$ such that $x^* = -\operatorname{grad}\log\phi(x)$, where $\phi(x) = \int_{V^*} e^{x\cdot y}dy$ is the characteristic function of V. It is known (see [2, 6]) that the $*$-function is a one-to-one mapping from V onto V^*. Let $G(V \to V^*)$ be the group of linear transformations mapping V onto V^*. A homogeneous cone V is said to be a domain of positivity if there is an element $S \in G(V \to V^*)$ so that S is symmetric and positive definite. It can be shown (see [6, 7]) that for a domain of positivity V, $x <_V y$ if and only if $y^* <_{V^*} x^*$.

In this note, we shall continue to consider the Riemann-Liouville operator

$$R_r f(x) = \int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)f(t)dt$$

and Weyl operator

$$W_r f(x) = \int_{\langle x,\infty\rangle} \Delta_V^{r-1}(t-x)f(t)dt$$

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on cones on R^d. It is worth noting that the Riemann-Liouville operators whose kernels are complex power functions associated with the cone V were extensively studied in [1], although they are different from the Riemann-Liouville operator we shall study here.

Theorem 1. *Let V be a domain of positivity in R^d. If $1 \le p \le q < \infty$, $r-1 > \sigma(V)$, and $\gamma < -\sigma(V)(1+\frac{q}{p'}) - \sigma(V^*) + q(\frac{1}{p} - r + 1) - 3$, then for any $f: V \to R^+$,*

$$(1)\quad \left(\int_V \Delta_V^{\gamma-q}(x)(R_r f(x))^q\,dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\right)^{1/p}.$$

We show (1) in the case $r \ge 1$ (see [7]). Since $-1 \le \sigma(V) \le 0$ for any cone V, Theorem 1 extends the result to the case $0 < r < 1$. It is also worth noting that $\Delta_V(x) \to 0$ as x approaches the boundary of V. Hence, the kernel of R_r approaches infinity as t approaches any pont on the boundary of $x - V$ in the case $r - 1 > \sigma(V)$. In the one dimensional case, where $V = (0, \infty)$, $\sigma(V) = -1$, $r > 0$, Theorem 1 gives the boundedness of the Riemann-Liouville operator on the half line.

In the proof of Theorem 1 we shall deal with integrals on the cone of the form

$$\int_{\langle 0,x\rangle} \Delta_V^\alpha(x-t)\Delta_V^\beta(t)dt$$

and

$$\int_{\langle x,\infty\rangle} \Delta_V^\alpha(t-x)\Delta_V^\beta(t)dt$$

where $\alpha, \beta < 0$. The following two lemmas prove that, under certain conditions on α and β, the integrals are finite for each $x \in V$ so that they can be "integrated" out. When restricted to the one dimensional case, these two lemmas give the best results.

Lemma 1. *Let V be a homogeneous cone and let*

$$g(x) = \int_{\langle 0,x\rangle} \Delta_V^\alpha(x-t)\Delta_V^\beta(t)dt, \qquad x \in V.$$

If $\alpha > \sigma(V)$ and $\beta > \sigma(V)$, then $g(x)$ is finite for each $x \in V$ and is homogeneous of order $\alpha + \beta + 1$. Hence, there is a constant c for which

$$g(x) = c\Delta_V^{\alpha+\beta+1}(x), \qquad x \in V.$$

Proof. Let $y \in V$. By Fubini's theorem, we have

$$\begin{aligned}\int_{\langle 0,y\rangle} g(x)dx &= \int_{\langle 0,y\rangle} \Delta_V^\beta(t)\left(\int_{\langle t,y\rangle} \Delta_V^\alpha(x-t)dx\right)dt\\ &= \int_{\langle 0,y\rangle} \Delta_V^\beta(t)\left(\int_{\langle 0,y-t\rangle} \Delta_V^\alpha(z)dz\right)dt\\ &\le \int_{\langle 0,y\rangle} \Delta_V^\beta(t)\left(\int_{\langle 0,y\rangle} \Delta_V^\alpha(z)dz\right)dt\\ &= \left(\int_{\langle 0,y\rangle} \Delta_V^\beta(t)dt\right)\cdot\left(\int_{\langle 0,y\rangle} \Delta_V^\alpha(z)dz\right).\end{aligned}$$

If $\alpha > \sigma(V)$ and $\beta > \sigma(V)$, the two integrals above are finite, and so $g(x)$ is finite for almost every $x \in V$.

Let $x_0 \in V$ be such that $g(x_0)$ is finite. Since V is homogeneous, for any $x \in V$ there exists $A \in G(V)$ so that $x = Ax_0$. Then we have

$$g(x) = g(Ax_0) = \int_{\langle 0, Ax_0\rangle} \Delta_V^\alpha(Ax_0 - t)\Delta_V^\beta(t)dt$$
$$= \int_{\langle 0, x_0\rangle} \Delta_V^\alpha(A(x_0 - z))\Delta_V^\beta(Az)|A|dz$$
$$= \int_{\langle 0, x_0\rangle} |A|^\alpha\Delta_V^\alpha(x_0 - z)|A|^\beta\Delta_V^\beta(z)|A|dz = |A|^{\alpha+\beta+1}g(x_0).$$

Hence, $g(x)$ is finite for each $x \in V$ and is homogeneous of order $\alpha + \beta + 1$. Therefore, there is a constant c for which $g(x) = c\Delta_V^{\alpha+\beta+1}(x)$, $x \in V$.

Lemma 2. *Let V be a domain of positivity and let*

$$h(x) = \int_{\langle x, \infty\rangle} \Delta_V^\alpha(t - x)\Delta_V^\beta(t)\,dt, \qquad x \in V.$$

If $\alpha > \sigma(V)$ and $\alpha + \beta < -3 - \sigma(V^*) - \sigma(V)$, then $h(x)$ is finite for each $x \in V$ and homogeneous of order $\alpha + \beta + 1$. hence, there is a constant c for which

$$h(x) = c\Delta_V^{\alpha+\beta+1}(x), \qquad x \in V.$$

Proof. The condition on $\alpha + \beta$ gives $-\sigma(V) > 3 + \sigma(V^*) + \alpha + \beta$. Let $\gamma \in R$ so that $-\sigma(V) > \gamma > 3 + \sigma(V^*) + \alpha + \beta$. We have, for $y \in V$,

$$\int_{\langle y, \infty\rangle} \Delta_V^{-\gamma}(x)h(x)\,dx$$
$$= \int_{\langle y, \infty\rangle} \Delta_V^{-\gamma}(x)\int_{\langle x, \infty\rangle} \Delta_V^\alpha(t - x)\Delta_V^\beta(t)\,dtdx$$
$$= \int_{\langle y, \infty\rangle}\left(\int_{\langle y, t\rangle} \Delta_V^\alpha(t - x)\Delta_V^{-\gamma}(x)dx\right)\Delta_V^\beta(t)\,dt$$
$$\le \int_{\langle y, \infty\rangle}\left(\int_{\langle 0, t\rangle} \Delta_V^\alpha(t - x)\Delta_V^{-\gamma}(x)dx\right)\Delta_V^\beta(t)\,dt.$$

Since $\alpha > \sigma(V)$ and $-\gamma > \sigma(V)$, by Lemma 1, we have

$$\int_{\langle y, \infty\rangle}\left(\int_{\langle 0, t\rangle} \Delta_V^\alpha(t - x)\Delta_V^{-\gamma}(x)dx\right)\Delta_V^\beta(t)dt = c\int_{\langle y, \infty\rangle} \Delta_V^{\alpha+\beta-\gamma+1}(t)dt.$$

Since V is a domain of positivity, a change of variable $t \to t^*$ gives

$$\int_{\langle y, \infty\rangle} \Delta_V^{\alpha+\beta-\gamma+1}(t)\,dt = \int_{\langle 0, y^*\rangle} \Delta_{V^*}^{-\alpha-\beta+\gamma-3}(t)\,dt.$$

Since $-\alpha - \beta + \gamma - 3 > \sigma(V^*)$, the last integral is finite. Hence, $h(x)$ is finite for almost every $x \in V$. Clearly, $h(Ax) = |A|^{\alpha+\beta+1}h(x)$ for $A \in G(V)$. Hence, $h(x)$ is finite for each $x \in V$ and is homogeneous of order $\alpha + \beta + 1$. Therefore, there is a constant c for which $h(x) = c\Delta_V^{\alpha+\beta+1}(x)$, $x \in V$.

Proof of Theorem 1. Noting the condition on γ in the hypothesis, we can choose b so that $\sigma(V) < b < (-3 - \sigma(V^*) - \sigma(V) - \gamma + q(\frac{1}{p} - r + 1))\frac{p'}{q}$.

Using Hölder's inequality, we have

$$\begin{aligned}
&\int_V \Delta^{\gamma-q}(x)(R_r f(x))^q\,dx \\
&\quad = \int_V \Delta^{\gamma-q}(x)\left(\int_{\langle 0,x\rangle} \Delta_V^{(r-1)/p}(x-t)f(t)\right. \\
&\qquad\qquad \left.\cdot \Delta_V^{-b/p'}(t)\Delta_V^{(r-1)/p'}(x-t)\Delta_V^{b/p'}(t)dt\right)^q dx \\
&\quad \le \int_V \Delta^{\gamma-q}(x)\left(\int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)f^p(t)\Delta_V^{-b(p-1)}(t)dt\right)^{q/p} \\
&\qquad\qquad \cdot\left(\int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)\Delta_V^{b}(t)\,dt\right)^{q/p'} dx.
\end{aligned}$$

Noting that $r-1 > \sigma(V)$ and $b > \sigma(V)$, by Lemma 1, we have

$$\begin{aligned}
&\int_V \Delta^{\gamma-q}(x)(R_r f(x))^q\,dx \\
&\quad \le c\int_V \Delta_V^{\gamma-q+(r+b)q/p'}(x)\left(\int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)f^p(t)\Delta_V^{-b(p-1)}(t)\,dt\right)^{q/p} dx.
\end{aligned}$$

Since $q/p \ge 1$, by the Minkowski integral inequality, we have

$$\begin{aligned}
&\int_V \Delta^{\gamma-q}(x)(R_r f(x))^q\,dx \\
&\quad \le c\left(\int_V f^p(t)\right. \\
&\qquad \left.\cdot\Delta_V^{-b(p-1)}(t)\left(\int_{\langle t,\infty\rangle} \Delta_V^{(r-1)q/p}(x-t)\Delta_V^{\gamma-q+(r+b)q/p'}(x)\,dx\right)^{p/q} dt\right)^{q/p}.
\end{aligned}$$

Noting that $(r-1)q/p > \sigma(V)$ and $(r-1)q/p + \gamma - q + (r+b)q/p' = rq - q/p + \gamma - q + bq/p' < -3 - \sigma(V^*) - \sigma(V)$, by Lemma 2, we have

$$\begin{aligned}
&\int_V \Delta^{\gamma-q}(x)(R_r f(x))^q\,dx \\
&\quad = c\left(\int_V f^p(t)\Delta_V^{-b(p-1)}(t)\Delta_V^{((r-1)q/p+\gamma-q+(r+b)q/p'+1)p/q}(t)\,dt\right)^{q/p} \\
&\quad = c\left(\int_V f^p(t)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(t)\,dt\right)^{q/p}.
\end{aligned}$$

Using Theorem 1 and the fact that Weyl's operator is the dual of Riemann-Liouville's operator, we can prove the following norm inequality for Weyl's operator on cones.

Theorem 2. *Let V be a domain of positivity in R^n. If $1 \le p \le q < \infty$, $r-1 > \sigma(V)$, and $\gamma > \sigma(V)(1+q/p') + \sigma(V^*)q/p' + q(1+2/p')$, then*

$$(2)\quad \left(\int_V \Delta_V^{\gamma-q}(x)(W_r f(x))^q dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\right)^{1/p}.$$

Now we consider the Hardy operator

$$Hf(x) = \int_{\langle 0,x\rangle} f(t)dt$$

on cones in R^d. As a corollary of the main theorem in [7], we have shown that if $1 \le p \le q \le \infty$ and $\gamma < -\sigma(V)q/p' - \sigma(V^*) + q/p - 2$, then

$$\left(\int_V \Delta_V^{\gamma-q}(x)(Hf(x))^q dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{(\gamma+1)p/q-1}(x)dx\right)^{1/p}.$$

It is natural to inquire whether there exist appropriate numbers α and β so that

$$(3)\qquad \left(\int_V \Delta^{\beta}(x)(Hf(x))^q dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta^{\alpha}(x)dx\right)^{1/p}$$

holds for all $f \ge 0$ when $1 \le q < p < \infty$. In the one dimensional case, the fact that (3) does not hold for any values of α and β when $1 \le q < p < \infty$ is simply a consequence of a theorem in [3] concerning the Hardy inequality with general weights. This result can be generalized to cones in R^d.

Theorem 3. *Let V be a domain of positivity in R^d. If $1 \le q < p < \infty$, then for any values α and β, there is no constant $c > 0$ such that* (3) *holds for all* $f \ge 0$.

Using the Hardy operator with weight, we see immediately that Theorem 3 is equivalent to the following theorem.

Theorem 4. *Let*

$$H_\alpha f(x) = \int_{\langle 0,x\rangle} f(t)\Delta_V^{\alpha}(t)dt,$$

and let V be a domain of positivity in R^d. If $1 \le q < p < \infty$, then for any values of α and β, there is no constant $c > 0$ such that

$$(4)\qquad \left(\int_V \Delta_V^{\beta}(x)(H_\alpha f(x))^q dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{\alpha}(x)dx\right)^{1/p}$$

holds for all $f \ge 0$.

Proof of Theorem 4. First we show that in order that a $c > 0$ exist for which (4) holds for all $f \ge 0$, α and β must satisfy $(\alpha+1)/p' + (\beta+1)/q = 0$ and $\beta < -1$.

Assume that (4) holds for some values of α and β. Then (4) implies that for all $z \in V$,

$$\left(\int_{\langle z,\infty\rangle} \Delta_V^{\beta}(x)\left(\int_{\langle 0,x\rangle} f(t)\Delta_V^{\alpha}(t)dt\right)^q dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{\alpha}(x)dx\right)^{1/p}.$$

Further, it implies

$$(5)\quad \left(\int_{\langle 0,z\rangle} f(t)\Delta_V^{\alpha}(t)dt\right)\cdot\left(\int_{\langle z,\infty\rangle}\Delta_V^{\beta}(x)dx\right)^{1/q}\le c\left(\int_V f^p(x)\Delta_V^{\alpha}(x)dx\right)^{1/p}.$$

Choose a sequence $\{V_n\}$ of nested cone intervals so that $\overline{V}_n\subset\langle 0,z\rangle$ and $V_n\nearrow\langle 0,z\rangle$. Note that $\int_{V_n}\Delta_V^{\alpha}(t)dt<\infty$. Let $f_n(x)=\chi_{V_n}(x)$. Substituting $f_n(x)$ in (5) we have

$$(6)\qquad \left(\int_{V_n}\Delta_V^{\alpha}(t)dt\right)^{1/p'}\cdot\left(\int_{\langle z,\infty\rangle}\Delta_V^{\beta}(x)dx\right)^{1/q}\le c.$$

It follows that $\int_{V_n}\Delta_V^{\alpha}(t)\,dt$ is bounded and so $\int_{\langle 0,z\rangle}\Delta_V^{\alpha}(x)dx$ is finite. It also follows from (6) that $\int_{\langle z,\infty\rangle}\Delta_V^{\beta}(x)\,dx$ is finite for each z. It is known [6] that

$$(7)\qquad \Delta_V(x)\le\rho|x|^d\quad\text{for some }\rho>0.$$

Therefore, in order that $\int_{\langle z,\infty\rangle}\Delta_V^{\beta}(x)\,dx$ be finite for each z it is necessary that $\beta<-1$.

Let $f(x)=\chi_{\langle 0,z\rangle}(x)$. Substituting $f(x)$ in (5) we have

$$(8)\qquad \left(\int_{\langle 0,z\rangle}\Delta_V^{\alpha}(t)dt\right)^{1/p'}\cdot\left(\int_{\langle z,\infty\rangle}\Delta_V^{\beta}(x)\,dx\right)^{1/q}\le c,\qquad z\in V.$$

Integrating these integrals in (8) and taking the supremum over $z\in V$, we have

$$\sup_{z\in V}\Delta_V^{(\alpha+1)/p'}(z)\Delta_V^{(\beta+1)/q}(z)\le c.$$

Therefore, it is necessary that $(\alpha+1)/p'+(\beta+1)/q=0$.

Next, we show that (4) cannot hold even if α and β satisfy the aforementioned conditions. First assume that $1<q$. Let $a=(q-1)/(p-q)$ and $b=q/(p-q)$. Note that $a>0$, $b>0$, and $(\alpha+1)(a+1/p)+(\beta+1)b=0$. Take $z_0\in V$ with $\Delta_V(z_0)=1$. Define

$$f_n(x)=\Delta_V^{a(\alpha+1)}(x)\min(n,\Delta_V^{b(\beta+1)}(x))\chi_{\langle 0,nz_0\rangle}(x),\qquad x\in V,\ n=1,2,\dots.$$

Clearly, for each n, $f_n^p(x)$ is integrable on V. We show that $\int_V f_n^p(x)\,dx\to\infty$ as $n\to\infty$.

Choose α_n so that

$$\Delta_V^{b(\beta+1)}(\alpha_n z_0)=\alpha_n^{db(\beta+1)}=n.$$

Then we have

$$\begin{aligned}\int_V f_n^p(x)\Delta_V^{\alpha}(x)\,dx&\\ \ge\int_{\langle\alpha_n z_0,nz_0\rangle}&\Delta_V^{a(\alpha+1)p}(x)\cdot\Delta_V^{b(\beta+1)p}(x)\Delta_V^{\alpha}(x)\,dx\\ =\int_{\langle\alpha_n z_0,nz_0\rangle}&\Delta_V^{-1}(x)\,dx.\end{aligned}$$

Since $\beta + 1 < 0$, it follows that $\alpha_n \to 0$ and $\langle \alpha_n z_0, n z_0 \rangle \nearrow V$ as $n \to \infty$. Noting that $\Delta_V^{-1}(x)$ is not integrable on V, we have that

$$\int_V f_n^p(x)\Delta_V^\alpha(x)\,dx \to \infty \quad \text{as } n \to \infty.$$

Using Fubini's theorem and noting that $\int_{\langle x,\infty\rangle} \Delta_V^\beta(y)\,dy$ is finite for every $x \in V$, we have

$$\begin{aligned}
&\int_V \left(\int_{\langle 0,y\rangle} f_n(z)\Delta_V^\alpha(z)dz\right)^q \Delta_V^\beta(y)\,dy \\
&= \int_V \left(\int_{\langle 0,y\rangle} f_n(x)\Delta_V^\alpha(x) \left(\int_{\langle 0,y\rangle} f_n(z)\Delta_V^\alpha(z)dz\right)^{q-1} dx\right) \Delta_V^\beta(y)\,dy \\
&\ge \int_V \left(\int_{\langle 0,y\rangle} f_n(x)\Delta_V^\alpha(x) \left(\int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)dz\right)^{q-1} dx\right) \Delta_V^\beta(y)\,dy \qquad (9) \\
&= \int_V f_n(x)\Delta_V^\alpha(x) \left(\int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)dz\right)^{q-1} \left(\int_{\langle x,\infty\rangle} \Delta_V^\beta(y)dy\right) dx \\
&= c\int_V f_n(x)\Delta_V^\alpha(x) \left(\int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)dz\right)^{q-1} \Delta_V^{\beta+1}(x)\,dx,
\end{aligned}$$

where c is a constant independent of f_n.

Since $\beta + 1 < 0$, $f_n(x)\Delta_V^{-a(\alpha+1)}(x)$ is a decreasing function of $x \in V$ in the partial ordering defined by V. Further, we have that

$$\begin{aligned}
&\int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)dz \\
&= \int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)\Delta_V^{-a(\alpha+1)}(z)\Delta_V^{a(\alpha+1)}(z)\,dz \qquad (10) \\
&\ge f_n(x)\Delta_V^{-a(\alpha+1)}(x)\int_{\langle 0,x\rangle} \Delta_V^{a(\alpha+1)+\alpha}(z)\,dz.
\end{aligned}$$

By (7), in order that $\int_{\langle 0,x\rangle} \Delta_V^\alpha(z)\,dz$ be finite, it is necessary that $\alpha > -1$. Thus, $a(\alpha+1) > 0$ and $\int_{\langle 0,x\rangle} \Delta_V^{a(\alpha+1)+\alpha}(z)\,dz$ is finite. So the last integral in (10) equals $c\Delta_V^{(\alpha+1)(a+1)}(x)$ and

$$\int_{\langle 0,x\rangle} f_n(z)\Delta_V^\alpha(z)dz \ge c f_n(x)\Delta_V^{\alpha+1}(x),$$

where c does not depend on f_n.

Therefore, (9) becomes

$$\begin{aligned}\int_V \left(\int_{\langle 0, y\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right)^q \Delta_V^{\beta}(y)dy \\ \geq c \int_V f_n(x)\Delta_V^{\alpha}(x) \left(\int_{\langle 0, x\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right)^{q-1} \Delta_V^{\beta+1}(x)dx \\ \geq c \int_V f_n(x)\Delta_V^{\alpha}(x) f_n^{q-1}(x)\Delta_V^{(\alpha+1)(q-1)}(x)\Delta_V^{\beta+1}(x)\, dx.\end{aligned}$$

Noting that

$$\Delta_V^{(\alpha+1)(q-1)+(\beta+1)}(x) \geq f_n^{p-q}(x),$$

we finally have

$$\int_V \left(\int_{\langle 0, y\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right)^q \Delta_V^{\beta}(y)dy \geq c \int_V f_n^p(x)\Delta_V^{\alpha}(x)dx.$$

Therefore,

$$\left(\int_V \left(\int_{\langle 0, y\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right)^q \Delta_V^{\beta}(y)dy \right)^{1/q} \geq c \left(\int_V f_n^p(x)\Delta_V^{\alpha}(x)dx \right)^{1/q},$$

where c is independent of $f_n(x)$. Since $\int_V f_n^p(x)\Delta_V^{\alpha}(x)dx \to \infty$ as $n \to \infty$ and $q < p$, there is no constant c such that for all f_n,

$$\left(\int_V \left(\int_{\langle 0, y\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right)^q \Delta_V^{\beta}(y)dy \right)^{1/q} \leq c \left(\int_V f_n^p(x)\Delta_V^{\alpha}(x)dx \right)^{1/p}.$$

So we proved Theorem 4 in the case $1 < q$. If $q = 1$, we define

$$f_n(x) = \min(n, \Delta_V^{b(\beta+1)}(x))\chi_{\langle 0, nz_0\rangle}(x), \qquad x \in V, \ n = 1, 2, \dots,$$

and (9) becomes the following simple inequality:

$$\begin{aligned}\int_V \left(\int_{\langle 0, y\rangle} f_n(z)\Delta_V^{\alpha}(z)dz \right) \Delta_V^{\beta}(y)dy \\ = \int_V f_n(x)\Delta_V^{\alpha}(x) \left(\int_{\langle x, \infty\rangle} \Delta_V^{\beta}(y)dy \right) dx \\ = c \int_V f_n(x)\Delta_V^{\alpha+\beta+1}(x)dx \geq c \int_V f_n^p(x)\Delta_V^{\alpha}(x)\, dx.\end{aligned}$$

The theorem is proved.

Now we consider the Hardy operator of the form

$$\widetilde{H}_\alpha f(x) = \int_{\langle x, \infty\rangle} f(t)\Delta_V^{\alpha}(t)\, dt.$$

For $\widetilde{H}_\alpha$ we expect the following similar result:

Theorem 5. *Let V be a domain of positivity in R^d. If $1 \leq q < p < \infty$, then for any values of α and β, there is no constant $c > 0$ such that*

$$\left(\int_V \Delta_V^\beta(x)(\widetilde{H}_\alpha f(x))^q dx\right)^{1/q} \leq c\left(\int_V f^p(x)\Delta_V^\alpha(x)dx\right)^{1/p} \tag{11}$$

holds for all $f \geq 0$.

Proof. Assume that for some α and β there is a constant $c > 0$ such that (11) holds for all $f \geq 0$. Let $g \geq 0$ be a function defined on V with $\int_V g^p(y)\Delta_V^\alpha(y)dy = 1$. Then, for $f \geq 0$,

$$\begin{aligned}
&\int_V \left(\int_{\langle 0,y\rangle} f(x)\Delta_V^\beta(x)dx\right) g(y)\Delta_V^\alpha(y)\,dy \\
&= \int_V \left(\int_{\langle x,\infty\rangle} g(y)\Delta_V^\alpha(y)\,dy\right) f(x)\Delta_V^\beta(x)\,dx \\
&\leq \left(\int_V \left(\int_{\langle x,\infty\rangle} g(y)\Delta_V^\alpha(y)\,dy\right)^q \Delta_V^\beta(x)\,dx\right)^{1/q} \left(\int_V f^{q'}(y)\Delta_V^\beta(y)\,dy\right)^{1/q'} \\
&\leq c\left(\int_V g^p(y)\Delta_V^\alpha(y)dy\right)^{1/p} \left(\int_V f^{q'}(y)\Delta_V^\beta(y)\,dy\right)^{1/q'} \\
&= c\left(\int_V f^{q'}(y)\Delta_V^\beta(y)dy\right)^{1/q'}.
\end{aligned}$$

Thus, for $f \geq 0$,

$$\left(\int_V \Delta_V^\alpha(y)\left(\int_{\langle 0,y\rangle} f(x)\Delta_V^\beta(x)dx\right)^{p'} dy\right)^{1/p'} \leq c\left(\int_V f^{q'}(y)\Delta_V^\beta(y)dy\right)^{1/q'}.$$

But this is impossible by Theorem 4. So there are no α and β so that (11) holds for all $f \geq 0$.

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